Lecture 11

7.4 - Integration by Partial Fractions

Consider the integral $\int \sec x \, dx$. The trick used to compute it is a bit out of the blue...

Consider this:

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} dx \quad \left(\begin{array}{l} u = \sin x \\ du = \cos x dx \end{array} \right)$$

$$= \int \frac{1}{1-u^2} du$$

How do we integrate this? Notice:

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{1/2}{1+u} + \frac{1/2}{1-u} \qquad (\star)$$

So, $\displaystyle \int \frac{1}{1-u^2} du = \frac{1}{2} \left(\int \frac{1}{1+u} du + \int \frac{1}{1-u} du \right)$

$$= \frac{1}{2} \left(\ln|1+u| - \ln|1-u| \right) + C$$

$$= \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2} \ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \ln\left|\sqrt{\frac{(1+\sin x)^2}{\cos^2 x}}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

The procedure in (☆) is called <u>Partial Fraction decomposition</u>.

Fundamental Theorem of Algebra

Every polynomial can be factored into linear factors of the form $ax+b$ and irreducible quadratic factors of the form ax^2+bx+c (irred. means $b^2-4ac<0$).

<u>Def</u>: A <u>rational function</u> is a quotient of two polynomials

$$\frac{P(x)}{Q(x)}.$$

The goal of this section is to integrate rational functions. The procedure to integrate them has up to 3 steps:

1) If the degree of $P(x)$ is larger than that of $Q(x)$

2) Unless the rational part can be integrated in other ways (u-sub, trig. sub., etc.), use partial fractions to decompose it

3) Integrate.

The method of partial fractions will decompose a proper rational function $\frac{R(x)}{Q(x)}$ $(\deg R < \deg Q)$ as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^k} \quad \text{and/or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

where A & B are constants and $k>0$.

We already know how to integrate these types of expressions.

We will go through increasingly more complex examples:

$\boxed{1.}$ $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x+b_1)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n}$$

Multiply through by the LCD (i.e. $Q(x)$):

$$R(x) = A_1(a_2x+b_2)\cdots(a_nx+b_n) + A_2(a_1x+b_1)(a_3x+b_3)\cdots(a_nx+b_n)$$
$$+ A_n(a_1x+b_1)\cdots(a_{n-1}x+b_{n-1})$$

Equating coefficients of powers of x, we can solve for $A_1, \cdots, A_n$.

Ex: Evaluate $\int \frac{1}{x^2-25}\,dx$

$\boxed{2.}$ $Q(x)$ has repeated linear factors, i.e., $(a_i x+b_i)^k$, $k>1$.

The idea is similar, but the difference is for every factor of the form $(a_i x+b_i)^k$ that shows up, we include a sum of the form $\frac{A_1}{a_i x+b_i}+\frac{A_2}{(a_i x+b_i)^2}+\cdots+\frac{A_k}{(a_i x+b_i)^k}$

For example, the decomposition of:

$$\frac{x^3+2x+2}{(x-2)^3(x-1)^2}=$$

Ex: Compute $\int \frac{2x+3}{x^3+2x^2+x}\,dx$

$\boxed{3.}$ $Q(x)$ has irreducible quadratic factors, none of which are repeated. For every factor $a_ix^2+b_ix+c_i$ we include a term $\frac{A_ix+B_i}{a_ix^2+b_ix+c_i}$.

For example, the decomposition of:

$$\frac{x^2+x+1}{x(x^2+1)} =$$

Ex: Evaluate $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx$